Problem 14

Use indirect reasoning to prove that $\log_2 5$ is an irrational number.

Solution

Suppose that $\log_2 5$ is a rational number, that is, a ratio of two positive irreducible integers, m and n.

$$\log_2 5 = \frac{m}{n}$$

$$2^{m/n} = 5$$

$$(2^{m/n})^n = (5)^n$$

$$2^m = 5^n$$

$$2(2^{m-1}) = \underbrace{5 \cdot 5 \cdots 5 \cdot 5}_{n \text{ times}}$$

The left side is an even number because it can be written as 2 times an integer 2^{m-1}. The right side, however, does not have a factor of 2 in its prime factorization, so it is in fact an odd number. This is a contradiction. Therefore, $\log_2 5$ is an irrational number.